Other theorems of a similar nature could be added. These theorems may be illustrated by the following simple equations, the numbers in the parentheses indicating the theorems which the equations are intended to illustrate. $y^2 - 1 = x^3 - 3x$, $y^2 - 1 = x^3$(1, 2, 3, 4); $y^2 - 2 = x^3 - 3x$, $y^2 + 2 = x^3 - 3x$ (5); $y^2 = x^3$ (6); $y^2 = x^4 - x^6$ (7).

It is, of course, true that the advantages of this method of constructing graphs are greatest for the more complicated forms of equations.

This principle may also be applied in the construction of graphs of parametric equations. For, if the parametric equations are $x = \varphi(t)$, $y = \psi(t)$ we may use as our auxiliary equations $x = \varphi(y')$, $y = \psi(x')$ and proceed as before.

UNDERGRADUATE MATHEMATICS CLUBS.

Edited by R. C. Archibald, Brown University, Providence, R. I.

In so far as information is obtainable the December issue of this Monthly will contain a directory of undergraduate mathematics clubs which have been in operation during at least a part of 1917–18. Certain statistics, and general discussion of club interests, will be introduced also. It is hoped that those clubs which have either failed to discover themselves to the editor or neglected to reply to his repeated solicitations for information, will report to him as soon as possible and, in any case, not later than October 6.

CLUB ACTIVITIES.

Mathematics Club of Iowa State Teachers College, Cedar Falls, Iowa.

This club was organized in December, 1909, "to further the work in mathematics in the college and in the state." The officers for 1917–18 were: President, Ira S. Condit, head of the department of ma hematics; secretary, Irma Hemphill '18; executive committee, Professor Peter Luteyn and Grace Hillier '18, who prepare programs and decide on dates of meetings The programs between June, 1916, and March 1918, were as follows:

June 28, 1916: "Teaching factoring" by Professor Robert D. Dougherty.

July 12: "Standard tests" by Professor E. E. Watson, of Parsons College.

August 2: "The problem and the process of analysis in arithmetic" by Professor Condit.

August 16: "Ratio and proportion," by Professor Charles W. Wester of Iowa State College and Rodney W. Babcock, instructor in mathematics at the University of Wisconsin.

October 18: "The place of mathematics in education" by Professor Wester.

November 15: Report of Committee on Elimination, of Iowa State Teachers Association, by Ruth Smith '18 and Tracey Hodson '18.

April 24, 1917: "Methods of statistical study" by Professor Wester.

May 16: Conservation in mathematics—"Conservation of time and energy by the development of algebraic symbolism" by Irma Hemphill '18; "The

metric system" by Alice O'Connor '18; "Short cuts in arithmetic" by Mae Howell '18; "Short cuts in high school mathematics" by Doris de Bar '18; "Conservation in geometry" by Ruth Smith '18.
July 11: "Elimination in mathematics" by Professor Emma L. Lambert.
July 25: "Elimination in mathematics" by Professor Luteyn.
March 6, 1918: "Contents of a course in arithmetic" by Professor Wester.

JUNIOR MATHEMATICS CLUB, University of Minnesota, Minneapolis, Minn.

The first meeting of this club was held on December 9, 1914. All students who have completed the course in integral calculus, and seniors who are majoring in mathematics, are eligible for membership. The staff of instruction in the department of mathematics and astronomy is also invited to attend the meetings of the club. The number of its members was 20 last year and the average attendance at meetings was 15.

Officers, 1917–18: President, Jemima Olson '18; secretary, Elizabeth Carlson Gr. These officers constitute the program committee also.
November 8, 1917: "The drawing of a circle having five times the area of a given circle" by Jemima Olson '18.
November 22: "How to draw a straight line without the use of a ruler" by Adelina Anderson '18.
December 6: "The Russian peasant method of multiplication" by Elma F. Hario '18.
December 13: Joint meeting with the University Mathematical Club[1]—"A generalized form of mathematical induction" by Professor William H. Bussey; "A note on the eight queens problem"[2] by Professor Bussey.
January 17, 1918: "The reduction of a quadratic form to the sum of squares" by Vera Wright Gr.
March 7: "Magic squares" by Hildegarde Swenson '18.
March 28: "Wallace lines" by Roger A. Johnson, Professor of Mathematics at Hamline University, St. Paul, Minn.
April 10: "Calculating machines" by Ralph M. Barton, instructor in mathematics, and Professor Bussey.
April 18: Annual banquet.
April 24: "The educational value of the history of mathematics" by William O. Beal, assistant astronomer to the university.

MATHEMATICS CLUB OF MOUNT HOLYOKE COLLEGE, South Hadley, Mass.

This club, which has just completed the tenth year of its existence, was founded for the purpose of "presenting to its members a broader view of mathematics." Last year it had 38 active and 15 associate members, the average attendance at

[1] This club is composed of faculty members and graduate students. Its programs are usually of too advanced a character for members of the Junior Mathematics Club.

[2] *Cf.* W. W. R. Ball, *Mathematical Recreations and Essays*, fifth edition, London, Macmillan, 1911, pp. 113–118.

meetings being 20. Junior and senior students who are doing major work in mathematics are eligible for active membership and such students become members by signing the constitution. Any sophomore who has elected mathematics as one of her major subjects may become an associate member by a majority vote of the members present at any meeting. Any member of the staff of the mathematics department and any graduate student in mathematics is eligible for either active or associate membership. Only active members may vote or hold office. Active members are charged a tax of fifty cents a year.

Officers 1917–18: President, Alice Weeks '18; vice-president, Evelyn Clift '19; secretary-treasurer, Jeannette Bickford '18; executive board: Ruth Carpenter '18, and Professor Emilie N. Martin.

October 21, 1916: Social meeting; the club was entertained by guessing various mathematical puzzles.

November 17: "Mathematical theory of probabilities and its application to games of chance" by Ruth Carpenter '18; "Magic squares" by Alice Weeks '18.

December 16: "Poincaré" by Helen Hughes '17; "Fermat, Fermat's Theorem, and Painlevé" by Florence Allen '17; an article on Sonja Kovalevski read by Ethel Anderson '19.

January 16, 1917: A talk on the models of surfaces owned by the college, and an exhibition of photographs of snow crystals by Professor Martin.

February 17: "Linkages" by Margaret Wilcox '19 and Evelyn Clift '19 (illustrated by models which they had constructed for the occasion); selections from Cassius Keyser's "Essays and Addresses" [1] read by Ruth Hemenway.

March 24: "Continued fractions" by Jeannette Bickford '18.

April 21: "Photogrammetry and its use in the present war" by Charles L. Bouton, associate professor of mathematics at Harvard University.

May 22: Election of officers for 1917–18.

In 1917–18 the number of meetings was reduced to five on account of war-work in which the members were engaged.

September, 1917: Social meeting to receive new members.

November: "Training in mathematics in a Russian artillery school" by Mr. Alexander Pell of South Hadley.

February 16, 1918: "Mathematics in the present war" by Captain Peter F. Field, professor at the University of Michigan.

March 9: "Applied mathematics in the laboratory work of a great electric company" by Agnes Eastman '13.

May 25: "Mathematics of insurance" by Helen Hughes '17; election of officers for 1918–19.

THE MATHEMATICAL CLUB OF THE UNIVERSITY OF NEBRASKA, Lincoln, Neb.

This club was organized in 1915 "to stimulate the mathematical interests of its members by presenting illuminating and varied phases of mathematics in

[1] The selections were taken from two essays: "The human worth of rigorous thinking" and "The human significance of mathematics."

pure and in applied form." According to the preliminary organization circular: "Many absorbing topics that can hardly be dealt with in the class room will find a natural place in this club. It is hoped that the members will gain a new and valuable insight into the beauty and wealth of the mathematical realm and a greater knowledge of the many ways in which mathematics touches life."

The charter members were recommended by the faculty. Thereafter members have been elected by the club on the recommendation of its executive committee which, as a matter of established custom, first consults the faculty. "As a rule the faculty recommends to membership those students who have shown distinction in their first year mathematics courses (through analytic geometry)"

Officers, 1917–18: President, Julia L. Torrence '18; vice-president, Eimo G. Funke '19; secretary-treasurer, Frances Botkin '19; faculty-adviser, Professor Henry Blumberg. These officers constituted the executive committee.

The club meets once a month, 7.30–9 00 p.m.; the number of members in attendance has varied from 20 to 60, with an average of about 35. All members are expected to contribute to the topics of the evening. Interesting problems are posted on the club's bulletin board and these problems are later discussed at the meetings. During 1916–17 three prizes were awarded by the club, mainly for activity in connection with the bulletin board. The programs for 1915–18 have been as follows:

October 14, 1915: Organization meeting; talks by different members of the faculty.

November 11: "Various definitions of π" by James H. Taylor '16; "The game of nim" by Professor Blumberg.

December 9: "Some problems from analysis situs" by Edward M. Kadlack '16; "Selected facts from the theory of numbers" by Professor William C. Brenke.

January 13, 1916: Social meeting, log fire, refreshments, everybody expected to bring an anecdote about a famous mathematician.

February 10: "Cantor's famous contribution to the study of the infinite" by Professor Blumberg; "The cycloid" by Herbert Grumman '15.

March 9: "Euclid's fifth postulate" by Olive Bayles '16; "Soap bubbles and mathematics" by Professor Oliver Gish.

April 13: "Lightning calculators" by Ezra Andreson '18; "The method of Archimedes" by Professor Albert L. Candy.

May 11: "Some unsolved problems in mathematics" by Professor Ellery W. Davis; social hour.

October 12: Election of officers; social hour.

November 9: "Summation of certain series" by Walter F. Weiland '18; "History of perspective drawing" by Professor Blumberg.

December 14: "Certain problems of interpolation" by Professor W. C. Brenke; bulletin board problems.

January 11, 1917: Social meeting; mathematical games; everybody expected to bring a geometric, algebraic or logical paradox.

February 15: "History of logarithms" by William F. Joachim '18.

March 15: "Selected topics from the theory of probability" by Professor Davis; bulletin board problems.
April 15: "The roots of unity" by Alva L. Sikes '19; "History of the Arabic numerals" by Professor Candy.
October: Election of officers; social hour.
November 9: "Selected topics from projective geometry" by Frances Botkin '19; "Mathematical curricula in different countries" by Professor Candy.
December 13: "Magic squares" by Josefa Seely '18; "Graphical methods" by Professor Lulu Runge.
March 14, 1918: Bulletin board problems; social hour with mathematical games.
April 11: "Selected facts from the theory of numbers" by Professor Blumberg; bulletin board problems; mathematical games.
May 9: "Development of mathematical symbols" by Bernice Downing '18; bulletin board problems; social hour with mathematical games and mathematical conundrums.

THE MATHEMATICS CLUB OF THE UNIVERSITY OF OKLAHOMA, Norman, Okla.

At a meeting on October 26, 1916, a group of students, who had decided to organize a mathematics club, appointed a committee, with Professor Samuel W. Reaves, head of the department, as chairman, to draw up a constitution. This constitution, adopted at the following meeting, was as follows:

PREAMBLE.

We, the undersigned, appreciating the advantages to be derived from an association which shall give opportunity for the presentation and discussion of mathematical subjects of interest, do hereby organize ourselves into a mathematical club, and we agree to be governed by the following Constitution.

ARTICLE I. *Name.*

Section 1. This association shall be called The Mathematics Club of the University of Oklahoma.

ARTICLE II. *Members.*

Section 1. Membership in this Club is limited to such students and teachers in the University of Oklahoma as are interested in the subject of mathematics.

Section 2. Proposals for membership shall be in writing and may be submitted at any regular meeting.

Section 3. Voting upon members shall be by ballot, a majority vote being necessary for election.

ARTICLE III. *Officers.*

Section 1. The officers of this club shall be a President, a Vice-President, and a Secretary-Treasurer, which officers shall be elected from among the students majoring in mathematics.

Section 2. Officers of the club shall be elected by ballot at the first regular meeting of each semester.

Section 3. Each officer shall serve until his successor is duly elected. Vacancies may be filled temporarily by presidential appointment until the next regular meeting of the club.

Section 4. The three officers mentioned in Section 1, together with one faculty member selected by the faculty members of the club, shall constitute the Program Committee.

ARTICLE IV. *Meetings.*

Section 1. Regular meetings of the club shall be held on the second and fourth Thursdays of each month.

ARTICLE V. *Miscellaneous.*

Section 1. This club shall have the power to make rules for its meetings, levy assessments upon its members, and perform other acts not inconsistent with this constitution.

Section 2. One-third of all the members of the club shall constitute a quorum.

Section 3. Amendments to this constitution shall be offered in writing, shall lie upon the table for two weeks, and shall require for adoption a two-thirds vote of all members present.

Section 4. In all cases not otherwise provided for this club shall be governed by "Roberts's *Rules of Order*" as parliamentary guide.

During 1916–17 the following meetings (with an average attendance of 15 members) were held:

November 9, 1916: "How we have learned to count" by Dr. Nathan Altshiller, instructor in mathematics.

December 14: "Finger counting" by Harold Gimeno '18; "Finger calculation" by Enoch B. Ferrell '18.

January 16, 1917: "Mathematical wrinkles"[1] by Thomas L. Sorey '18 and Hugh S. Lieber '18.

February 13: "Some properties of triangles" by Professor Reaves.

March 13: "Paper folding" by Ella Mansfield '18.

March 27: "Life of Descartes" by Earl Bonham '17.

April 10: "Invention of logarithms" by Professor Edmund P. R. Duval; "If at the beginning of our era one cent had been placed at four per cent compound interest, what would be the radius of the gold sphere equivalent to the capital accumulated to date?" by Margaret Coleman '17.

April 17: "Mathematics and astronomy" by Professor Harry C. Gossard.

"Owing to the fact that most of the advanced students in mathematics, including both presidents of the club (Thomas L. Sorey '18 and Enoch B. Ferrell '18), entered military service, while the others left college, to fill the vacancies created by the military draft, it was decided to suspend the activities of the club for the year 1917–18."

TOPICS FOR CLUB PROGRAMS.

12. Geometry of Four Dimensions.

By HENRY P. MANNING, Brown University.

If but a single meeting is to be devoted to this subject the most interesting way to take it up is to give a simple statement of theorems and proofs after the manner in which plane and solid geometry is studied in the schools. Most of those who belong to mathematics clubs have heard tales of strange things that are to be found in hyperspace, and what they would like to know is the geometry itself and the mysterious paths by which these things are found. Probably the best topics to take up would be perpendicularity and various kinds of angles, and then, if there is time, some of the theorems about parallel lines, planes and hyperplanes.[2]

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[1] A book entitled *Mathematical wrinkles for teachers and private learners* was published by Samuel I. Jones at Gunter, Texas, in 1912.

[2] A list of theorems as I have taken them up on two or three occasions is given in *The Mathematics Teacher*, Vol. 7, Dec., 1914, pp. 49–58.

Such a treatment ought to be the central or main part of any study of this subject, but if there is time to look at it from other points of view the following topics (arranged under 7 headings) may be considered.[1]

1. The history of the idea of a fourth dimension. It will be convenient to think of three periods: down to 1827, from 1827 to 1870, from 1870 to the present day.

In the writings of Aristotle (384–322 B.C.), Pappus (end of the third century), and, according to Simplicius (sixth century A.D.), of Ptolemy (about 150 A.D.) there are speculations about the number of dimensions of space and the possibility or impossibility of there being more than three. The same matter is referred to by the English philosopher Henry Moore (1614–1687), and by Kant (1724–1804).[2]

In the development of algebra, which at first ascribed a geometrical meaning to all of its terms, there came about a reluctant extension of these terms beyond those which describe figures of two and three dimensions.[3]

Finally, there is a suggestion by d'Alembert (1717–1783), and after him by Lagrange (1736–1813), that time might be regarded as the fourth dimension.[4]

In the second period certain leading mathematicians took up particular parts of hypergeometry and proved theorems or developed formulæ.[5] These developments were mostly by analytical methods, but sometimes synthetic methods were used, and in them all the geometrical conceptions were made quite clear. There was also, however, much work done in pure analysis with equations and algebraic forms in n variables, where the language of geometry is sometimes used, but the conceptions of geometry are almost entirely absent.[6]

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[1] Most of the material for this work will be found in the author's *Geometry of Four Dimensions*, Macmillan, New York, 1914, and in a book of fourth dimension essays, *The Fourth Dimension Simply Explained*, Munn, New York, 1910. These will be referred to as *Geometry* and *Essays*.

[2] Pappus, after speaking of the three-line and four-line locus (see Heath's *Apollonius*, Cambridge, 1896, Introduction, chap. v) says that we cannot say of more than six lines, "the ratio of the content of four to that which is contained by the rest, since there is nothing contained by more than three distances. Men a little before our time have allowed themselves to interpret such things, signifying nothing at all comprehensible, speaking of the product of that which is contained by such lines into the square of this or the content of those. These things might, however, be stated and shown generally by means of compound ratios"—Hultsch ed., Vol. II, Berlin, 1877, p. 680. For other references see *Geometry*, pp. 1–3.

[3] *Geometry*, pp. 2–3.

[4] *Geometry*, p. 4; R. C. Archibald, "Time as a Fourth Dimension," *Bulletin of the American Mathematical Society*, Vol. 20, 1914, p. 409.

[5] Möbius, *Der barycentrische Calcul*, Leipzig, 1827, § 140, p. 184; Cayley, "Sur quelques théorèmes de la géométrie de position," *Crelle's Journal*, Vol. 31, pp. 213–226, in particular, pp. 217–218; Sylvester, "On the Centre of Gravity of a Truncated Triangular Pyramid," *Philosophical Magazine*, fourth series, Vol. 26, Sept., 1863, pp. 167–183 or *Mathematical Papers*, Vol. II, No. 65; Clifford, *Educational Times*, Jan., 1866, or *Mathematical Reprints*, Vol. 6, pp. 83–87, or *Mathematical Papers*, London, 1882, p. 601.

[6] Green, *Mathematical Papers*, edited by N. M. Ferrers, London, 1871, p. 188; C. G. J. Jacobi, "De binis quibuslibet functionibus homogeneis," *Crelle's Journal*, Vol. 12, 1834, p. 1; Cayley, two papers published in the *Cambridge Mathematical Journal*, Vol. 3, 1841, or *Mathematical Papers*, Vol. I, Nos. 2 and 3; Schläfli, (1) "Ueber das Minimum des Integrals," etc., *Crelle's Journal*, Vol. 43, 1852, pp. 23–36; (2) "On the Integral," etc., *Quarterly Journal*, Vols. 2 and 3, 1858–1860.

For further details see *Geometry*, pp. 4–8.

Since 1870 we have witnessed a very rapid growth of the subject. References are given in the *Geometry* to Veronese, Halphen, Jordan, Stringham, Hathaway, Cole (p. 142), Keyser, Craig, and others. Nearly every meeting of a mathematical society has some paper, and nearly every volume of a mathematical journal contains some article bearing on this subject.

2. The many curious phenomena in space of four dimensions make a description of this subject especially interesting, even to those who do not care for mathematics. An account of these phenomena after the manner of the numerous "non-mathematical" articles that have been written would furnish an entertaining paper for a mathematics club. It is easy to write such a paper because almost everything in this geometry is analogous to things of plane and solid geometry. In fact, the mathematician in developing and establishing its theorems is all the time consciously or unconsciously using these analogies. Even the proofs are often analogous to the proofs of the lower geometries, although the analogy itself is never a proof.[1]

3. The analogy with two-dimensional beings has been used very freely to show our relation to four-dimensional space.[2] In particular, this idea has been developed in the form of a romance by E. A. Abbott, in *Flatland,* and by C. H. Hinton in *An Episode of Flatland.* A report on these books is sometimes made the subject in a meeting of a mathematics club. The former has also been dramatized.[3]

The question whether anyone is able, or can acquire the ability, to form a conception of space of four dimensions may depend partly on what we mean by the word conception. It is certainly possible by practice to acquire a facility in describing figures of such a space and in stating and proving the theorems of this geometry.[4]

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[1] The following are some of the phenomena referred to. They are explained many times in the *Essays* and in all articles of a similar nature. The new perpendicular direction, *Essays,* pp. 45, 71, 82, 122. Passing out of a closed room, *Essays,* pp. 116, 123, 138, 149, 180, 188. Making symmetrical figures coincide by turning one of them over through hyperspace, *Essays,* pp. 48, 76, 95, 139, 158, 213. Turning a flexible hollow shell inside out, *Essays,* pp. 49, 170, 249. Knots, links of a chain, *Essays,* pp. 50, 145. Planes meeting only in a point, going around a plane, *Essays,* pp. 23, 24, 49. The hypercube, *Essays,* pp. 46, 72, 88, 92, 113, 147. Cutting apart a polyhedroid and spreading it out in a hyperplane, *Essays,* p. 74, *Geometry,* pp. 68, 238, 240, 248. Some of these subjects are mentioned by Professor Newcomb, "Philosophy of Hyperspace" (presidential address), *Bulletin of the American Mathematical Society,* Vol. 4, Feb., 1898, pp. 187–195; "Fairy land of Geometry," *Harper's Magazine,* Vol. 104, Jan., 1902, pp. 249–252, reprinted in *Sidelights on Astronomy,* New York, Harper and Brothers, 1906, pp. 155–164.

There is a story by Charles Johnstone, "Jones and the Fourth Dimension," *Harper's Magazine,* Vol. 129, June, 1914, pp. 117–122, that is rather amusing, though not quite accurate in its representation of a lack of distance in space of four dimensions.

[2] Essays, pp. 78, 81, 96, 115, 127, 155.

[3] Abbott, *Flatland,* London, 1884, Little, Brown and Company, Boston, 1907. Hinton, *An Episode of Flatland,* London, Swan, Sonnenschein and Co., 1907. In the *Mathematical Gazette,* Vol. 7, Jan., 1914, pp. 228–231, is an account of a dramatic performance of *Flatland* by a girls' school in Acton, England. This is reprinted in *School Science and Mathematics,* Vol. 14, Oct., 1914, pp. 583–587.

[4] See discussion of this question by C. J. Keyser, "Mathematical Emancipations," *Monist,* Vol. 16, 1906, pp. 65–83, particularly, pp. 81–82; reprinted in *The Human Worth of Rigorous Thinking,* New York, Columbia University Press, 1916, pp. 101–121. See also *Geometry,* pp. 15–16 and the last footnote on p. 9.

4. If the various topics suggested are taken up and several meetings are devoted to the geometry of four dimensions, the treatment of the geometry itself by synthetic methods as explained at the beginning of this article should come at this point.

5. The analytical geometry of four dimensions with formulæ for distances and angles of various kinds, the equations of hyperplanes, planes and lines, and a brief account of the hypersurfaces of the second degree, the hyperconicoids.[1]

6. Some particularly puzzling subjects may be suggested for further study.

Kinematics and mechanics of four dimensions promises to be very interesting. The cross section of a rope would be a sphere, not a circle. It would have to be fastened to a spherical ring, not to a circular ring. A chain could not be made of links, but alternate links and hollow spheres could be put together to form a chain. Instead of a two-dimensional wheel and a one-dimensional axle we must have a three-dimensional wheel, or a two-dimensional axis.[2]

If we lived on the three-dimensional boundary of a hypersphere, just as we actually live on the earth, held to it by an attraction towards the center, we could move in three directions. We might study the problem of changing pressure into motion (like that of the locomotive), or the problem of constructing a mill that will saw a hypersolid block into two pieces, just as we saw a log into boards.

Surfaces in hyperspace is another subject that needs to be studied a great deal. A curve in any kind of space extends only forwards and backwards and seems very much the same as in a plane as long as we confine our attention to the curve itself. It is only when we consider the space about the curve that we find in space of three dimensions the phenomena of knots and of torsion. So surfaces, considered by themselves, are very much the same in any space. As long as we confine our attention to the surface itself we find it two-dimensional and we move about on it very much as we do on a simple plane. There is no difficulty in understanding at least a small portion of an ordinary surface in any kind of space. But when we investigate the space about the surface, if the space is of more than three dimensions, we begin to be mystified. Rotation on an axis-plane and the

[1] This treatment is taken up by Jouffret, *Géométrie à quatre dimensions*, Paris, 1903, and (for the case of any number of dimensions) by Schoute, *Mehrdimensionale Geometrie*, *Sammlung Schubert*, XXXV and XXXVI, Leipzig, 1902 and 1905, I, § 6, pp. 125–180. Certain parts are worked out also in the following articles: F. N. Cole, "On Rotations in Space of Four Dimensions," *American Journal of Mathematics*, Vol. 12, 1890, pp. 191–210; T. Craig, "Displacements Depending on One, Two and Three Parameters in a Space of Four Dimensions," *American Journal of Mathematics*, Vol. 20, 1898, pp. 135–156; J. G. Hardy, "Curves of Triple Curvature," *American Journal of Mathematics*, Vol. 24, 1902, pp. 13–38; C. J. Keyser, "Concerning the Angles and the Angular Determination of Planes in 4-Space," *Bulletin of the American Mathematical Society*, Vol. 8, 1902, pp. 324–329. Two articles in the AMERICAN MATHEMATICAL MONTHLY may be mentioned here: B. H. Brown, "Centres of Similitude and their N-Dimensional Analogies," Vol. 23, May, 1916, pp. 155–159, and M. H. Sznyter, "Some Metrical Properties of the Pentahedroid in a Space of Four Dimensions," Vol. 24, Mar., 1917, pp. 113–119.

[2] See *Essays*, p. 31, Hinton, *The Fourth Dimension*, Swan, Sonnenschein and Co., London, 1904, pp. 31 and 71–73. The nature of rotations is taken up briefly at the end of an article by Beltrami, "Formules fundamentales de cinematique dans les espaces de courbure constante," *Bulletin des sciences mathématiques*, Vol. 11, 1876, pp. 233–240, *Opere*, Milan, Vol. III, 1911, pp. 23–29, and very fully by Cole, *loc. cit.*

fact that we can go around a surface just as in ordinary space we go around a line make one of its mysteries. The consideration of an entire surface that does not lie in any one three-dimensional space also involves peculiarities of hyperspace. An example is the surface of double revolution, a surface on which the ordinary plane geometry of Euclid holds true if we take for lines certain systems of circles. The study of surfaces is important in what is called analysis situs.[1]

7. The object and value of the study of this subject may be considered from three points of view: any direct applications that it may have, its relation to the science of geometry and to mathematics in general, and any philosophical discussions that it may involve.

We can use it to give simple proofs of certain theorems of ordinary geometry, such as the equivalence of symmetrical figures,[2] and theorems about certain figures that are sections or projections of simpler figures of hyperspace.[3] New figures have also been discovered in this way. Thus application has been made to architecture.[4] In mechanics it has been found useful to represent time as a fourth dimension. In this way the theory of relativity finds its simplest form of expression.[5] It can be used sometimes to give us graphical solutions of problems[6]. It helps us to visualize certain complicated groups of objects, such as determinants of more than three dimensions.[7] There are many interpretations of this geometry in which some other figure than the point is taken as element. These interpretations have been put forward to help us understand the geometry, but the relation can be reversed and the geometry applied to the study of these figures.[8] Its most important direct application lies in its furnishing names and concrete terms for certain complicated ideas and expressions of analysis.

In the next place we get a broader view of the nature of geometry by a study of geometry of four dimensions, and, as far as we can carry it, of higher dimensions. There are many theories in geometry that we hardly notice, if at all, in studying only plane and solid geometry. We understand better the foundations of geometry and we get an invaluable training in mathematical reasoning.[9]

Finally, the study of this geometry throws light on certain questions of philosophy, as, for example, the nature of space itself. The fact that we are able to develop other kinds of geometry besides the plane and solid geometry of our experience is put forward as disproving the theories of Kant on the subjective nature of our geometrical intuitions.

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[1] See *Geometry*, p. 219 and the reference there to Poincaré.

[2] *Geometry*, p. 149.

[3] See *Geometry*, p. 5, and the references there to Cayley and Veronese.

[4] Claude Bragdon, *Projective Ornament*, Manas Press, Rochester, N. Y., 1915.

[5] *Geometry*, p. 11.

[6] An example is Clifford's solution of a problem in probability referred to in *Geometry*, p. 5, and in footnote above.

[7] See abstract of paper by L. H. Rice, "Determinants of Many Dimensions," *Bulletin of the American Mathematical Society*, Vol. 23, 1916, p. 69.

[8] *Geometry*, p. 10.

[9] *Geometry*, p. 14.